Critical issues in the design-by-testing of glass structures

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Abstract

To assess the reliability of glass structures, a common practice is to test full-scale prototypes in the lab, and verify that the failure load is higher than that predicted from the design strength by means of structural calculations. However, any procedure of design-by-testing should be considered with great care because the gross strength of glass, being governed by the opening of pre-existing cracks on the material surface, strongly depends upon the type of defectiveness, the specimen size, the load history and the type of stress field (uniaxial, bi-axial). A model based upon an assumed law of subcritical crack propagation and a distribution *à la* Weibull of pre-existing flaws is presented. This allows to correlate the expected glass strength with the target probability of failure for any type of specimen size and load history. The discussion of paradigmatic examples confirms that appropriate theoretical considerations are needed for the correct interpretation of experimental results.

Keywords: Glass strength, design-by-testing, failure probability, fracture mechanics, subcritical crack growth, Weibull distribution.

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1. Introduction

The incessant investigation of ever greater transparency has led to an increasingly strong demand for glazed surfaces in modern construction works. Glass is being used in challenging elements, such as larger and larger panels, roofs, beams, floors, stairs and frames, where the brittle material is required to carry substantial loads, therefore achieving a definite "structural" role. Improvements in production and technologies, such as tempering, increase the macroscopic strength of this material; lamination of glass plies sandwiching polymeric interlayers mitigates the effect of brittleness, because the shards remain adherent to polymeric interlayers after glass breakage. Considerable research is being undertaken to improve the understanding of the load-carrying capacity of structural glass elements under the actions those elements are exposed to during their service life, in order to achieve the requirements in terms of safety and serviceability that are prescribed by construction standards.

The reliability of a structural design depends on the capability to determine the material failure strength with accuracy. The most used methods to measure the mechanical strength of glass are the Four Point Bending (4BP) test and the Coaxial Double Ring (CDR) test, which are precisely defined by harmonized standards [1, 2, 3]. Both tests induce a uniform stress field in the loaded area of the specimen: the 4PB test generates an almost uniaxial stress field[1], while in the CDR test the stress field is approximately equi-biaxial. Results are often interpreted using a two-parameter Weibull distribution [5], which is traditionally considered the best statistical approach [6]. The characteristic value of strength is defined as the 5% fractile of the population of data, and represents a reference quantity to be considered in the design.

But the strength of glass, the brittle material *par* excellence, is affected by some peculiar aspects, which are of minor importance in other building materials such as steel and concrete,

[1]The stress field is in general not perfectly uniaxial, because a stress concentration occurs in proximity of the edges, where defectiveness is in generally greater that in the core of the specimen [4]. Therefore, the results may be strongly influenced by the type of edge working.

but acquire a crucial role in this case. At the macroscopic level, glass does not exhibit any ductility and breaks as soon as the stress at a point overcomes a certain limit, but no theory of glass strength can disregards consideration of the underlying microstructure. In fact, the material strength is governed by the presence of existing microscopic surface flaws, which open and progress under the applied stress [7]. Therefore, Linear Elastic Fracture Mechanics (LEFM) is the most useful tool to investigate the mechanical property of glass and interpret its brittle character.

Surface treatments (especially along the edges) have a strong influence on the strength because they may alter the size and distribution of surface flaws, and the larger the surface, the higher is the probability of finding critical defects (size effect). The state of stress is also important, because cracks open in mode I and the probability of finding a dominant crack at right angle to the maximal tensile stress is higher under a equi-biaxial state of stress than under a uniaxial state of stress. An even more peculiar aspect is that cracks can slowly grow in time without any variation of the applied macroscopic stress. This phenomenon, usually referred to as *slow crack propagation* or *static fatigue* [8], makes the glass strength strongly dependent upon the load history.

This is why the results obtained from the standardized tests, where specimen size and load rate are prescribed, need to be re-scaled according to a theoretical model of crack growth before being interpreted statistically [9]. In other words, the characteristic value of strength must be referred to standard conditions, i.e., a particular specimen size (usually 1 square meter), a precise load rate (2 MPa *per* second) and a prescribed state of stress (equi-biaxial). Experimental data are broadly dispersed and strongly affected by several factors (surface treatments, border finishing) that influence the distribution of flaws on the surface. Moreover, the values associated with a prescribed fractile of the population of data considerably change if the specimens size, the load rate and the type of stress field are different from those taken as reference.

A common practice, used by many designers and also implemented in standards[2], is to produce full-scale prototypes to be tested in laboratory to determine whether their actual response meets the design requirements. Due to the costs of the prototypes, their number is necessarily low, and it is not rare to find structural calculations where the designers considers the results from testing of just one prototype[3]. In general, designers are happy if the prototype breaks at a stress level higher than the design strength, usually associated with the 5% fractile value of the assumed distribution of material strengths. Sometimes, designers strongly remark that the ultimate stress measured on the prototype is much higher than the design strength of glass prescribed by standards, arguing that such a value is too much on the safe side. However, some critical issues are neglected in this rationale. First of all, the characteristic value of strength is in general associated with the 5% fractile of the population of data, whereas when testing just a few specimens one should expect, albeit tentatively, results closer to the median, i.e., the value corresponding to the 50% probability of failure. Moreover, the size of the prototype and the complex state of stress to which it is subjected should be properly taken into account. Last but not least, the loading rate during the experiments affects the results because of the static fatigue phenomenon.

The aim of this article is to show how all the aforementioned aspects can affect the result of experimental investigations. Given a distribution of strength *à la* Weibull, whose parameters have been calibrated out of an extensive experimental campaign [4], and assumed a widely accepted model of slow-crack propagation [8], we consider the hypothetical testing of a reference structure. For this, we theoretically calculate the values of strength associated with a target probability of failure, taking into account the size-effect and the type of stress, supposing that loads are either constant in time, or applied at a constant rate. Three paradigmatic examples are presented: *i*) a plate under an equi-biaxial stress field, similar

[2]For example, large façade panels are usually tested according to standards [10, 11].

[3]This observation is the result of the experience of one of the authors while serving as a reviewer of plans of glass construction works. Such an experience has been mainly made at the Board of Public Works at the Ministry of Infrastructure and Transport of the Italian Republic.

in type to the configuration assumed in a CDR test according to the standard [3]; *ii*) a rectangular specimen in a four-point bending setup, chosen for the wide use that the 4PB test has in the practice to determine the flexural strength of beams and floors [2]; *iii*) an edge supported plate under uniform distributed load, as the paradigmatic representation of a façade panel exposed to wind pressure.

We will show that the failure loads associated with the 5% fractile and the median values for these different-in-type structures can be very different one-another, even assuming the same statistical distribution of strength for glass. Therefore, before designing an experimental test on a complex structure, it is necessary to preliminary estimate the consequences of size effect, state of stress, static fatigue, surface defectiveness, load rate, and define from this analysis the actual expectations in terms of structural strength for the required target probability. The method of analysis proposed in this article can take into account all these effects and, although applied here to three cases only, it can be extended to the most general configurations.

2. Probabilistic model of glass strength

The macroscopic mechanical properties of glass stem from its brittle nature, which is characterized by a high sensitivity to stress concentrations often caused by surface flaws. Accurate characterization of the fracture strength of glass must then incorporate the nature and response of such surface cracks, whose size and orientation are often unknown. Therefore, a probabilistic model needs to be used to statistically interpret the generally broadly dispersed experimental data.

2.1. Mechanical behavior of a single surface flaw: the subcritical crack growth model

Linear elastic fracture mechanics (LEFM) provides a good model to interpret the brittle failure of glass. This is caused by propagation of one dominant crack in mode I, while the contributions in mode II and mode III are always supposed to be negligible. Assuming that the dominant surface crack is semielliptical and orthogonal to the crack surface, the stress intensity factor in mode I is given by

$$K_I = Y \sigma_\perp \sqrt{\pi c}, \tag{2.1}$$

where $\sigma_\perp$ is the tensile stress normal to the crack plane, c is the size of the crack (size of the smallest of the two elliptical axes) and Y is a factor that takes into account the aspect ratio of the ellipse (for a semicircular crack $Y = 2.24/\pi$). Instantaneous failure of glass occurs when the stress intensity factor K_I exceeds a critical value, known as the *critical stress intensity factor* K_{Ic}. This can be considered as a material constant, related with the fracture toughness of the glass itself. The value $K_{Ic} = 0.75$ MPa m$^{0.5}$ can be conveniently used for practical purposes [12].

It is highly recognized that flaws can grow in size over time when they are exposed to a positive crack opening stress [7]. This phenomenon is often referred to as *subcritical crack growth* or *static fatigue*, because the glass element may fail after a certain time even when the crack size is far from the critical limit c_c, which from (2.1) is given by

$$c_c = \left(\frac{K_{Ic}}{Y f_c \sqrt{\pi}} \right)^2, \tag{2.2}$$

where f_c is the measured (macroscopic) stress at right angle to the crack plane in the case of *instantaneous* collapse.

It is often assumed [13, 14] that the speed of the subcritical crack growth can be expressed as a function of the stress intensity factor according to a power-law of the type [15]

$$\frac{dc}{dt} = v_0 \left(\frac{K_I}{K_{Ic}} \right)^n, \tag{2.3}$$

where v_0 and n are material constants that depend upon the thermo-hygrometric conditions and the type of glass. For soda-lime glass, one can conveniently assume $v_0 = 0.0013$ m/s for test performed at $40-70\%$ relative humidity, but such value may increase if the humidity is higher, so that it is customary to prudentially consider $v_0 = 0.0025$ m/s. Also the exponent n is influenced by the humidity and may vary between 12 and 20; however, there is agreement to consider $n = 16$, associated with RH $= 100\%$, as a representative value [16].

Considering a load history $\sigma_\perp = \sigma(t)$, integration of (2.3) between the time $t = 0$, when the crack size is initially c_i, and the failure time t_f, when $c = c_c$, leads to

$$\int_{c_i}^{c_c} c^{-n/2} dc = \int_0^{t_f} v_0 \left(\frac{\sigma(t) Y \sqrt{\pi}}{K_{Ic}} \right)^n dt. \tag{2.4}$$

In general [2, 3], tests are performed at a constant stress rate $\dot{\sigma}_{test}$, so that $\sigma(t) = \dot{\sigma}_{test} t$. Assuming v_0 and n to be constant, and denoting with f_{test} the tensile strength measured at the end of the test, one finds that the initial size of the dominant crack is of the form

$$c_i = \left[\frac{n-2}{2} \frac{v_0}{n+1} \left(\frac{Y\sqrt{\pi}}{K_{Ic}} \right)^n \frac{f_{test}^{n+1}}{\dot{\sigma}_{test}} + \left(\frac{Y\, f_{test} \sqrt{\pi}}{K_{Ic}} \right)^{n-2} \right]^{\frac{2}{n-2}}. \tag{2.5}$$

Such a parameter represents an index of the defectiveness initially present in the glass specimen.

2.2. Random population of surface flaws. Influence of state of stress and size effect

Equation (2.4) describes the lifetime of a single crack. In general, a glass contains a large number of randomly-oriented surface flaws: because of its brittle nature, failure is assumed to occur when the first flaw reaches the critical size (*weakest link model*). For this reason, the Weibull distribution is usually considered the best choice to statically characterize the material strength [6]. The failure probability of a specimen, whose surface A is under the stress $\sigma_\perp$ acting at right angle to the dominant crack plane, can be written as [17]

$$P_f = 1 - \exp\left[- \int_A \left(\frac{\sigma_\perp}{\eta_0} \right)^m \right] dA, \tag{2.6}$$

where η_0, the reference strength, and m, the shape exponent, are the parameters of the Weibull distribution. High values of m indicate low dispersion of data, and this is representative of a homogeneous defectiveness of the sample population.

The previous expression can be simplified by assuming a homogeneous and isotropic defectiveness, i.e., there is not a preferred orientation and a preferred location for the dominant crack. Let σ_1 and σ_2 represent the principal component of the (macroscopic) stress acting on A, and indicate with ψ the angle that the direction of σ_1 forms with respect to the normal to the dominant crack plane. Then, $\sigma_\perp$ can be written as

$$\sigma_\perp = \sigma_1 \left[\cos^2(\psi) + r \sin^2(\psi)\right], \tag{2.7}$$

having defined $r = \sigma_2/\sigma_1$. Since there is an equal probability of finding a dominant crack of any orientation (isotropic defectiveness), one can consider the mean value of all possible orientations, and obtain, from (2.6), the expression

$$P_f = 1 - \exp\left\{ - \int_A \left[\left(\frac{\sigma_1}{\eta_0}\right)^m \frac{1}{\pi} \int_0^\pi \left(\cos^2(\psi) + r \sin^2(\psi)\right)^m d\psi \right] dA \right\}. \tag{2.8}$$

Following [18], one can further introduce the correction factor C, accounting for the particular state of stress, in the form

$$C = \left[\frac{2}{\pi} \int_0^{\pi/2} \left(\cos^2(\psi) + r \sin^2(\psi)\right)^m d\psi \right]^{1/m}, \tag{2.9}$$

so that equation (2.8) can be re-written as

$$P_f = 1 - \exp\left[- \int_A \left(\frac{C\sigma_1}{\eta_0}\right)^m dA \right] = 1 - \exp\left[-kA \left(\frac{\sigma_{\max}}{\eta_0}\right)^m \right], \tag{2.10}$$

where $\sigma_{\max}$ represents the maximum tensile stress in the loaded area and the quantity kA ($k \leq 1$), referred to as the *effective area* A_{eff}, is defined as

$$A_{\text{eff}} = kA = \frac{\int_A (C\sigma_1)^m dA}{(\sigma_{\max})^m}. \tag{2.11}$$

Expression (2.10) allows to compare, from a statistical point of view, the results obtained from different-in-type tests. In the case of an equibiaxial stress distribution, like in the test standard [3], one has $\sigma_1 = \sigma_2 = \sigma_{eqbiax}$ and, from (2.9), $C = 1$. Therefore, (2.10) reads

$$P_f = 1 - \exp\left[-\int_A \left(\frac{\sigma_{eqbiax}}{\eta_0}\right)^m dA\right]. \tag{2.12}$$

In the case of a uniaxial state of stress, like in the 4PB test of [2], one has in (2.7) $r = 0$ and, from (2.9), $C = C_{uniax} = \left[\frac{2}{\pi}\int_0^{\pi/2}(\cos\psi)^{2m}\, d\psi\right]^{1/m}$. The fact that $C_{uniax} < 1$ indicates that a specimen under uniaxial stress has a lower failure probability than the same sample under a biaxial stress field, due to the lower probability of finding a dominant crack at right angle to the maximal tensile stress.

If one wishes to compare the aforementioned two different test configurations, for specimens with identical defectiveness the probability of failure should be the same provided that the stress is properly re-scaled according to (2.10). This is accomplished provided that

$$\left[\int_A \left(\frac{\sigma_{eqbiax}}{\eta_0}\right)^m dA\right] = \left[\int_A \left(\frac{C_{uniax}\,\sigma_{uniax}}{\eta_0}\right)^m dA\right]. \tag{2.13}$$

In the simplest case in which the stress is homogeneous in the same loaded area, one has

$$\sigma_{uniax} = \frac{\sigma_{eqbiax}}{\left[\frac{2}{\pi}\int\limits_0^{\pi/2} (\cos\psi)^{2m}\, d\psi\right]^{1/m}}. \tag{2.14}$$

The meaning of this expression is that one expects to find higher strengths if the specimen is loaded in uniaxial mode rather than in biaxial mode.

Of course, (2.10) does account for the size effect, because the larger the area, the higher is the probability to find a dominant crack of prescribed size. In order to compare the failure probabilities obtained from different test geometries and make the results homogeneous and comparable, it is customary to refer to a homogeneous *equibiaxial* stress distribution acting over the unit area (UA) of 1 m^2. For this case, since $C = 1$, one has from (2.10)

$$P_f = 1 - \exp\left[-UA\left(\frac{\sigma_{eqb,UA}}{\eta_0}\right)^m\right]. \tag{2.15}$$

For a generic area and a generic stress field, the statistical distribution is given by (2.10). For specimens with the same defectiveness the probabilities of failure are the same provided that

$$\left[-UA\left(\frac{\sigma_{eqb,UA}}{\eta_0}\right)^m\right] = \left[-kA\left(\frac{\sigma_{\max}}{\eta_0}\right)^m\right], \tag{2.16}$$

which leads to the following relationship

$$\sigma_{\max} = \sigma_{eqb,UA}\left(\frac{UA}{kA}\right)^{1/m}. \tag{2.17}$$

This equation allows re-scaling the results of a generic test with respect to the reference configuration of an equibiaxial state acting on an unitary area.

2.3. *Influence of loading history*

2.3.1. *Constant stress*

In order to determine the effects of static fatigue, the starting point is equation (2.4). The first analysis considers the effects of a constant load that produces the stress $\sigma(t) = \sigma_a$ in the interval $[0, t_f]$, being t_f the time when failure occurs. Supposing that the thermo-hygrometric parameters remain constant and neglecting dynamical effects, the critical crack size c_f can be evaluated *via* (2.2) by setting $f_c = \sigma_a$. Integration of (2.4) between the time $t = 0$, when the crack size is $c = c_i$, and the failure time t_f, when $c = c_f$, reads

$$\frac{2}{n-2}\left(c_i^{\frac{2-n}{2}} - c_f^{\frac{2-n}{2}}\right) = v_0 \left(\frac{Y\sqrt{\pi}}{K_{Ic}}\right)^n \sigma_a^n\, t_f. \tag{2.18}$$

From this, denoting with K_i the stress intensity factor associated with the initial crack length c_i as *per*

$$c_i = \left(\frac{K_i}{Y\sigma_a\sqrt{\pi}}\right)^2, \tag{2.19}$$

one obtains the time to failure t_f in the form

$$t_f = \frac{2}{n-2}\frac{K_i^{2-n}}{\frac{v_0}{K_{Ic}^n}Y^2\sigma_a^2\pi}\left[1-\left(\frac{K_i}{K_{Ic}}\right)^{n-2}\right]. \tag{2.20}$$

This expression can be re-written in terms of stress. Recalling from (2.19) that $K_i = Y\sigma_a\sqrt{\pi c_i}$, and denoting with σ_{if} the stress that would produce instantaneous rupture, so that $K_{Ic} = Y\sigma_{if}\sqrt{\pi c_i}$, one obtains

$$t_f = \frac{2}{n-2}\frac{{K_{Ic}}^2}{v_0 Y^2\sigma_a^2\pi}\left[\left(\frac{\sigma_{if}}{\sigma_a}\right)^{n-2} - 1\right]. \tag{2.21}$$

It is necessary now to give a statistical characterization of this expression, in order to consider the probability of failure in lifetime predictions. From (2.10), the probability to find a dominant crack at right angle to the direction of the stress σ_{if} is given by

$$P_f = 1-\exp\left[-kA\left(\frac{\sigma_{if}}{\eta_0^*}\right)^{m^*}\right], \tag{2.22}$$

where η_0^* and m^* are the Weibull parameters referred to *instantaneous* failure. From this one finds

$$\sigma_{if} = \eta_0^*\left[\frac{1}{kA}\ln\frac{1}{1-P_f}\right]^{1/m^*}, \tag{2.23}$$

which can be substituted into (2.21) to obtain an expression that indicates the probability of failure in the time t_f when the stress σ_a is applied.

The parameters η_0^* and m^* can be obtain from the population of measures obtained in a standardized test at constant stress rate $\dot{\sigma}_{test}$ [2, 3]. In fact, let f_{test} represent the measured tensile strength in such a test, and let

$$P_f = 1 - \exp\left[-kA\left(\frac{f_{test}}{\eta_0}\right)^m\right], \tag{2.24}$$

represent the corresponding Weibull statistics, obtained by interpolating the experimental data. But, recalling that $c_i = \left(\frac{K_{Ic}}{Y\sigma_{if}\sqrt{\pi}}\right)^2$, substituting in (2.5) one obtains

$$\left(\frac{K_{Ic}}{Y\sigma_{if}\sqrt{\pi}}\right)^2 \simeq \left[\frac{n-2}{2}\frac{v_0}{n+1}\left(\frac{Y\sqrt{\pi}}{K_{Ic}}\right)^n \frac{f_{test}^{n+1}}{\dot{\sigma}_{test}}\right]^{\frac{2}{n-2}}, \tag{2.25}$$

where in (2.5) the second term on the r.h.s. has been neglected with respect to the first term (a-posteriori verification indicates this simplification is perfectly licit). Obtaining f_{test} from (2.25), substituting in (2.24), and equating the resulting failure probability with that corresponding to (2.22), one finally finds

$$m^* = \frac{n-2}{n+1}\,m\,, \tag{2.26}$$

$$\eta_0^* = \left\{\left[\frac{n-2}{2}\frac{v_0}{n+1}\left(\frac{Y\sqrt{\pi}}{K_{Ic}}\right)^2\frac{1}{\dot{\sigma}_{test}}\right]^{\frac{1}{n+1}}\eta_0\right\}^{\frac{n+1}{n-2}}. \tag{2.27}$$

Equations (2.21) and (2.23) will be applied to some paradigmatic cases (Section 3) in order to evaluate the failure time t_f for different values of the applied stress σ_a and for several target failure probabilities P_f.

2.3.2. *Constant stress rate*

It is also of interest to evaluate, from a statistical point of view, the effects of the speed of loading on the results of a test. Let f_{test} be the failure stress in a reference test at the stress rate $\dot{\sigma}_{test}$ and let, in general, f_g be the failure stress for another test on identical specimen, for which the stress rate is $\dot{\sigma}$. Since c_i is an intrinsic material parameter, representative of the defects that are initially present in the material and independent of the type of test, from (2.5) one can write

$$\frac{n-2}{2}\frac{v_0}{n+1}\left(\frac{Y\sqrt{\pi}}{K_{Ic}}\right)^n \frac{f_{test}^{n+1}}{\dot{\sigma}_{test}}+\left(\frac{Y\, f_{test}\sqrt{\pi}}{K_{Ic}}\right)^{n-2}=\frac{n-2}{2}\frac{v_0}{n+1}\left(\frac{Y\sqrt{\pi}}{K_{Ic}}\right)^n \frac{{f_g}^{n+1}}{\dot{\sigma}}+\left(\frac{Y\, f_g\sqrt{\pi}}{K_{Ic}}\right)^{n-2}. \tag{2.28}$$

The strength f_{test} can be related to the failure probability according to equation (2.24) and can therefore be written as

$$f_{test}=\eta_0\left[\frac{1}{kA}\ln\frac{1}{1-P_f}\right]^{1/m}. \tag{2.29}$$

Substituting in (2.28), it is then possible to evaluate the strength f_g as a function of the load rate $\dot{\sigma}$ for different values of the failure probability P_f. This relationship will be discussed in the examples of Section 3.

2.4. *Parameters of the statistical distribution*

The strength of glass is in general obtained through standardized tests under precise thermo-hygrometric conditions (temperature $T = 24 \pm 3$°C and relative humidity $RH = 50 \pm 10\%$) and at constant load rate ($\dot{\sigma} = 2$ MPa/s). In Europe, the most used test methods to determine the mechanical strength of glass are the Four Point Bending test (4PB) [2] and the Coaxial Double Ring test (CDR) [1, 3]. Both of them aim at inducing a homogenous state of stress within the loaded area of the specimen, uniaxial for the 4PB test and equi-biaxial in the CDR test.

An extensive experimental campaign was conducted at the *Stazione Sperimentale del Vetro* in Italy and the main results are reported in [4]. The size of the specimens and the experimental apparatus were slightly different from those prescribed by the European Standards, but results were re-scaled to take into account the effective area according to (2.17).

Table 1: Weibull parameters for 6 mm specimens obtained from the experimental tests. Reference unit area UA= 1 m^2 and load rate $\dot{\sigma} = 2$ MPa/s.

Stressed surface	**Shape modulus** m	**Reference strength** η_0 [MPa $mm^{2/m}$]
Tin	7.3	406
Air	5.4	1096

Parameters of the Weibull probabilistic model, obtained by rescaling the experimental results to an equi-biaxial state of stress acting on the unit area UA= 1 m^2 and load rate $\dot{\sigma} = 2$ MPa/s, are summarized in Table 1. Recall that during the float production process one side of the glass paste is directly in contact with the molten tin bath (tin side), while the other surface is directly exposed to air (air side). The diverse boundary induces a diverse surface defectiveness on the surfaces of the glass ply, and two different Weibull distributions should be used for the tin and the air sides. This distinction will be maintained in the following.

3. Case studies

Three examples are now considered:

i. a glass ply of 1 m^2 under an *equi-biaxial* state of stress;

ii. the four-point-bending (4PB) of a glass ply;

iii. a square glass panel under uniformly distributed load.

For every case the time to failure at constant stress, as well as the effect of the stress rate on collapse load, will be discussed from a statistical point of view.

3.1. Equi-biaxial state stress of stress

For a glass plate of area 1 m^2, consider a loading configuration that generates an equibiaxial stress state on its surface, i.e., $r = 1$ in (2.7) and $C = 1$ in (2.9). The assumed Weibull distribution of strengths is (2.15), where m and η_0 are taken from the experiments of [4] and recalled in Table 1. Recall that these parameters, as indicated in Section 2.4, are already referred to the unitary area UA=1 m^2 and to a standard load rate $\dot{\sigma} = 2$ MPa/s. For the reasons discussed in Section 2.4, a distinction is made between the tin-side and air-side values.

In order to investigate the expected time to failure under constant load, equation (2.21) will be evaluated taking into account the expression (2.23) for σ_{if}, with $k = 1$. All the relevant parameters for this analysis are summarized in Table 2. Figure 1 shows the time-to-failure vs. stress (in logarithmic scale) for different values of the target failure probability, distinguishing the tin-side (Figure 1(a)) from the air-side (Figure 1(b)).

To illustrate, if in the test a constant (equi-biaxial) state of stress is applied for 5 minutes, the graph of Figure 1(a) (tin-side) indicates that, for a failure probability of 5% (usually associated with the characteristic fractile value of strength), the expected failure stress is about 28 MPa. If the failure probability is $P_f = 50\%$, which is representative of the median strength, the failure stress increases to 42 MPa.

When testing a very reduced number of specimens, it is not possible to build up any statistics. If just 1-2 specimens are used, as is usually the case for full-scale testing, one could expect any value of strength because the number of data is certainly insufficient for any interpretation.

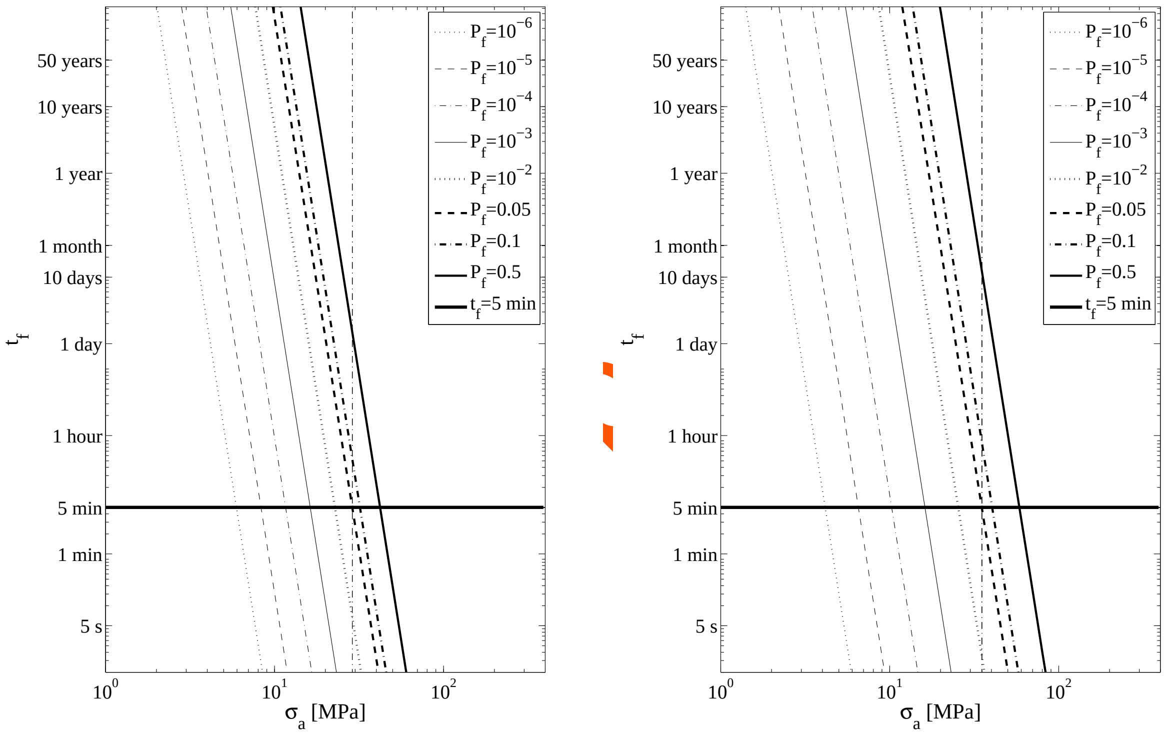
Table 2: Relevant parameters for the analysis of glass strength.

Thermo-hygrometric conditions	n	v_0 [m/s]	Y	K_{Ic} [MPa m$^{1/2}$]
T=23°C, RH=55%	16	0.0025	$2.24/\pi$	0.75

However, albeit tentatively, the value that should be expected to be the most frequent is the one associated with the median of the distribution of strengths.

It should be noticed from the graphs of Figure 1 that the median is about twice the 5% fractile value of the distribution, for both the tin-side and the air-side. Recall that the 5% fractile is usually taken as a characteristic value of strength and represents the reference value for structural calculations. Therefore, if the test provided the median value of strength one should not be surprised to find, from the experiment, a value much higher than the reference value for design. However, we have to mention, from private and public discussions with engineers and architects, that when tests provide strengths much higher than the reference stress, there is always a certain criticism for standards, considered too much conservative and imposing a redundant and expensive design. However, we have to observe that, whereas for more traditional building materials such as steel the difference between the characteristic and the median value of the assumed distribution of strength is not so relevant, for the case of glass it is quite impressive. This is essentially due to the higher dispersion of the strength data for this material.

Another consideration can be made. Referring to the graphs of Figure 1(a) and considering the strength associated with the 5% failure probability, a virtual vertical line can be drawn from it that intercepts the 50% failure probability line at a precise point. It may thus be observed that more than one day of applied constant stress is necessary to obtain the 50% of collapse at the same stress level that corresponds to a 5% failure probability in 5 minutes. However, one can obtain the 50% probability of failure in 5 minutes by increasing the stress level from 28 MPa to 42 MPa. Therefore, if the tests have to be accelerated, it is convenient to increase the applied stress level. The model provides the relationship between the state of stress, the time to failure and the target probability of failure.



(a) (b)

Figure 1: Time-to-failure vs. stress plots for different failure probabilities. Equi-biaxial state of stress: a) tin-side surface; b) air-side surface.

Analogous considerations, at the qualitative level, hold for the air-side of Figure 1(b). As remarked in [4], in general the strength on the air-side is slightly higher than on the tin-side. It should be observed that almost ten days are necessary to have a $P_f = 0.5$ probability of breaking the specimens under the same applied stress associated with a 5% failure probability

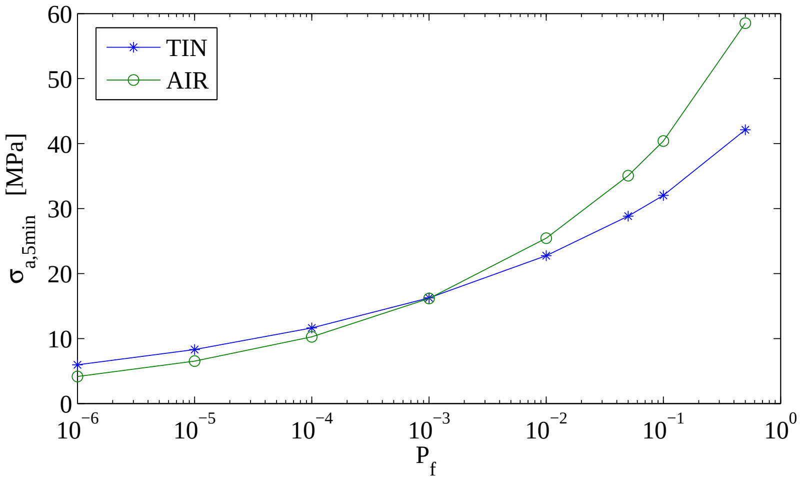


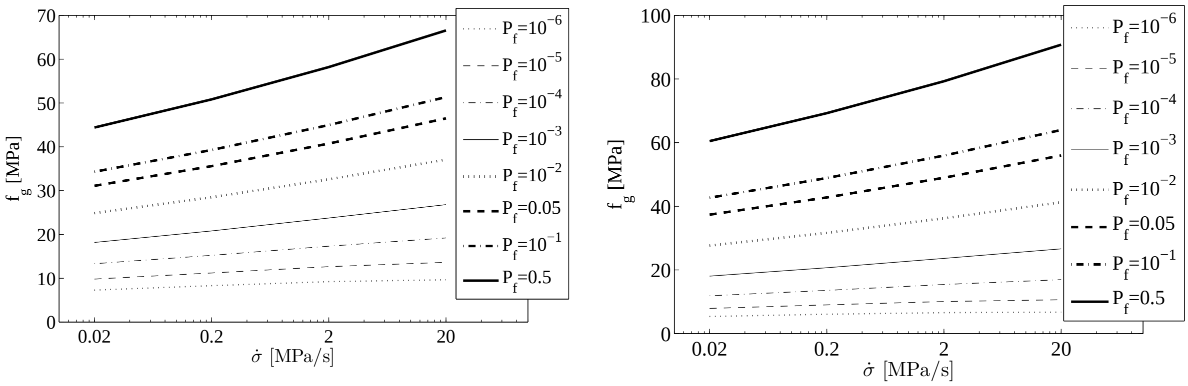
Figure 2: Strength corresponding to an equi-biaxial state of stress applied for 5 minutes, as a function of the failure probability. Tin- and air-side distributions.

in 5 minutes.

Figure 2 shows the comparison between the failure stresses under a 5-minute constant load for the tin-side and the air-side surfaces, as a function of the failure probability P_f. In general the air-side curve is higher than the tin-side curve apart from very low values of P_f. This is a consequence of the fact that, as observed in [4], although the air-side is stronger than the tin-side, the statistical dispersion of data is also higher: this penalizes the values associated with a low probability of collapse.

Let us pass to consider the case of a test at constant loading rate. The re-scaling of the expected results for different load rates is obtained from equation (2.28), taking into account (2.29).

The graphs of Figure 3 represent the failure stress f_g as a function of the loading rate $\dot{\sigma}$ (in logarithmic scale) for several values of the failure probability P_f, for both the tin-side (left-hand-side graph) and the air-side (right-hand-side graph). An increment of the strength can be observed when, fixing the stress rate (for example $\dot{\sigma} = 2$ MPa) one passes from $P_f = 0.05$ ($f_g \simeq 40$ MPa) to $P_f = 0.5$ ($f_g \simeq 58$ MPa). It should also be noted that the curves increase their slope as the failure probability increases, being almost horizontal for very low P_f. For



(a) (b)

Figure 3: Glass strength as a function of the loading rate for varying failure probability. Equi-biaxial stress state: a) tin-side surface; b) air-side surface.

example, for $P_f = 10^{-6}$ the expected value of strength is $f_g \simeq 8$ MPa for $\dot{\sigma} = 0.2$ MPa/s, whereas it is $f_g \simeq 9$ MPa for $\dot{\sigma} = 2$ MPa/s. On the other hand, considering the curve $P_f = 0.5$, one passes from $f_g \simeq 51$ MPa for $\dot{\sigma} = 0.2$ MPa/s, to $f_g \simeq 58$ MPa for $\dot{\sigma} = 2$ MPa/s.

In conclusion, increasing the loading rate does not affect very much the strength when low probabilities are taken into account, while it has a strong effect when one considers the probability associated with the median value of strength. This can be of importance while interpreting the results on a very reduced number of specimens, for which the median value is, in any case, the expected value.

3.2. Four point bending (4PB)

The four point bending is a widely used testing procedure to characterize the flexural strength of glass. For the sake of this example, reference is made to the tests recorded in [4], that only slightly differ from the prescriptions of EN 1288-3:2000 [2] for what concerns the specimen size. The support span is 360 mm, the load span 200 mm, the glass width is 400 mm and the thickness 6 mm. Therefore, a loaded area of 400×200 mm^2, corresponding to that part of the beam comprised between the supports, has been considered in the calculations. Such a type of test generates an almost uniaxial stress field on the surface of the glass[4] so that, now, the value of the correction factor C from equation (2.9) is ($\sigma_2 = 0$ and $r = 0$)

$$C = \left[\frac{2}{\pi} \int_0^{\pi/2} (\cos^2 \psi)^m d\psi \right]^{1/m} . \tag{3.1}$$

To evaluate the time to failure under constant stress, equations (2.21) and (2.23) can be applied. The value of kA (effective area), to be used in (2.23), is evaluated through (2.11). Since under 4PB the stress on the surface between the supports is constant and equal to the maximum stress, setting $\sigma_1 = \sigma_{\max}$ one finds

$$kA = \frac{\int_A (C\sigma_1)^m dA}{\sigma_{\max}^m} = C^m A \quad \to \quad k = C^m . \tag{3.2}$$

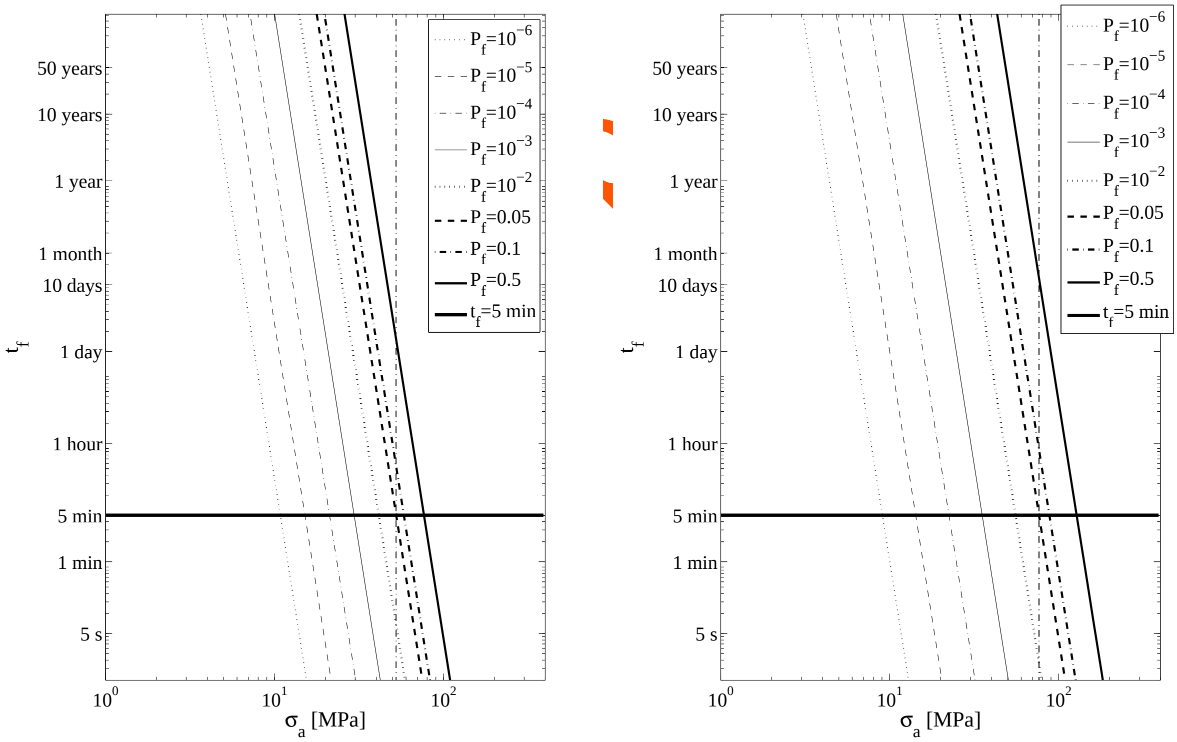
Values of k obtained for the tin-side and the air-side are summarized later on in Table 3.

Figure 4 shows the time-to-failure vs. stress graphs for the tin-side and the air-side surface for different values of P_f. Observe, first of all, that for any given P_f and t_f, the expected failure stress is higher than that corresponding to the equibiaxial stress field of Section 3.1. This is a general characteristic, confirmed by experiments [4], that is evidenced in the graphs of Figure 5, where the stress $\sigma_{a,5min}$ corresponding to $t_f = 5$ minutes, (values corresponding to the horizontal lines in Figure 4(a) and Figure 4(b)) is plotted as a function of P_f. In particular, the value corresponding to $P_f = 0.5$ is of the order of ~ 76 MPa (tin-side), much greater than the one associated with the equibiaxial stress field (~ 42 MPa, as *per* Figure 2). The higher values are due to the fact that, in a uniaxial state of stress, the probability that the orientation of the crack plane is orthogonal to the principal direction of stress is less

[4]We are neglecting, in this calculations, the edge effects due to the high width-on-thickness ratio, that lead to higher stress close to the borders than at the center. This effect is certainly of importance, but our aim here is to discuss, from a theoretical point of view, the effects of the type of stress, i.e., uniaxial vs. equi-biaxial.

than in a equi-biaxial state of stress. In other words, in a equi-biaxial stress field more crack orientations could be critical.

Moreover, observe that also in this case the air-side curve is always higher than the tin-side curve, apart from the region of very low probabilities of failure, where the vice-versa is true although the two curves almost coincide. As already recalled, this is due to the fact that the air-side is stronger than the tin-side, but its statistical dispersion of data is higher [4]. This penalizes the values associated with the air-side while considering very low probability of collapse, but the effect is less evident than for the equi-biaxial state of stress (see Figure 2). This is because, roughly speaking, the uniaxial type of the stress somehow mitigates the probabilistic dispersion.



(a) (b)

Figure 4: Time-to-failure versus stress plot for different failure probabilities. 4PB test: a) tin-side surface; b) air-side surface.

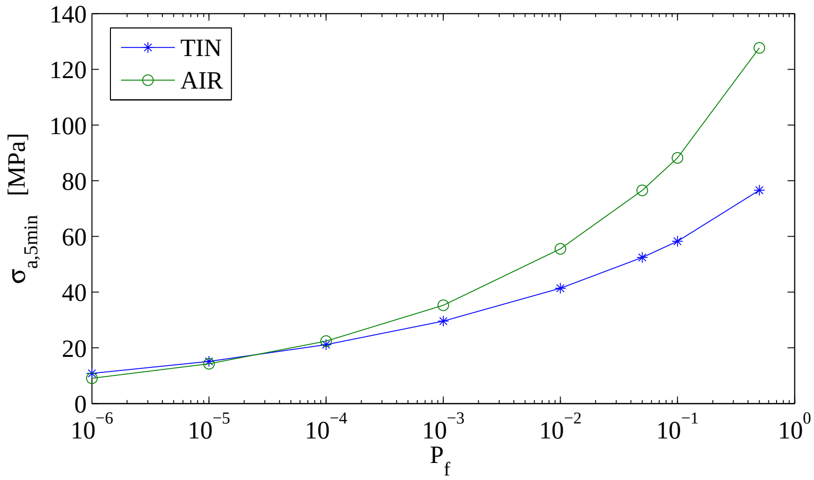


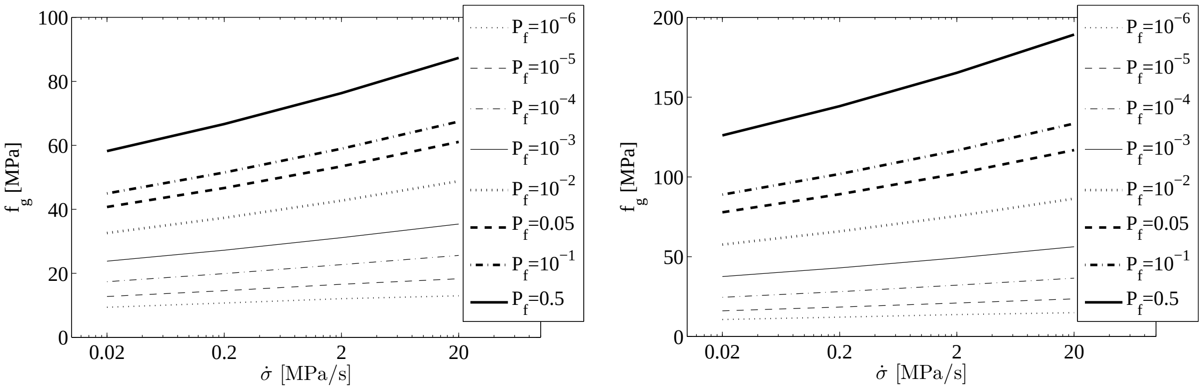
Figure 5: Four Point Bending (4PB) test. Strength corresponding to a load applied for 5 minutes as a function of the failure probability.

In the case of load histories at different stress rates $\dot{\sigma}$, the failure stress f_g can be probabilistically estimated through equations (2.28) and (2.29). Results are plotted in Figure 6 for both the tin- and the air-side. Similarly to the previous case, for any fixed load rate there is an increase in the failure stress for increasing probabilities of failure. In general, the load rate is uneventful in practice for low probabilities, while it is of importance for higher values. Moreover, the air-side presents much higher values of the failure stress than the tin side.

3.3. Plate under distributed load

Consider a 6 mm thick monolithic glass plate of area 1000×1000 mm^2, simply supported at the edges and under a uniformly distributed load. This case can be representative of a situation in which the plate is part of a façade under wind pressure.

In order to evaluate the effective area $A_{\text{eff}} = kA$, the domain representative of the plate surface is divided into N small sub-elements of equal area ΔA_i, $i = 1...n$, and for the i-*th* element the mean value of the principal stress components $\sigma_{1,i}$ and $\sigma_{2,i}$, and consequently the ratio $r_i = \sigma_{2,i}/\sigma_{1,i}$, is evaluated from a finite element analysis, considering geometric second



(a) (b)

Figure 6: Glass strength as a function of the loading rate for varying failure probability. 4PB test: a) tin-side surface; b) air-side surface.

order effects. The correction factor C_i of each element ΔA_i is thus calculated through (2.9). Then, from (2.10), the probability that the plate fails under the applied load can be approximated through the expression

$$P = 1 - \exp\left[- \sum_{i=1}^{N} \left(\frac{C_i \, \sigma_{1,i}}{\eta_0} \right)^m \Delta A_i \right] , \tag{3.3}$$

so that the counterpart of (2.11) becomes

$$k = \frac{\sum_{i=1}^{N} (C_i \, \sigma_{1,i})^m \Delta A_i}{A(\sigma_{\max})^m} . \tag{3.4}$$

In this way, one obtains [19] $k_{tin} = 0.138$ for the tin-side and $k_{air} = 0.1764$ for the air-side. For the sake of comparison, the values of k for all the cases so far considered are summarized in Table 3. As expected, the plate under uniform pressure corresponds to the lower values of k, indicating that for this condition the state of stress is the least "dangerous".

Once k has been calculated, expressions (2.21) and (2.23) can be applied to find the time-to-failure t_f as a function of the applied (constant) maximum stress. Results are plotted in Figure 7(a) and Figure 7(b) for the tin-side and the air-side respectively. Figure 8 shows the

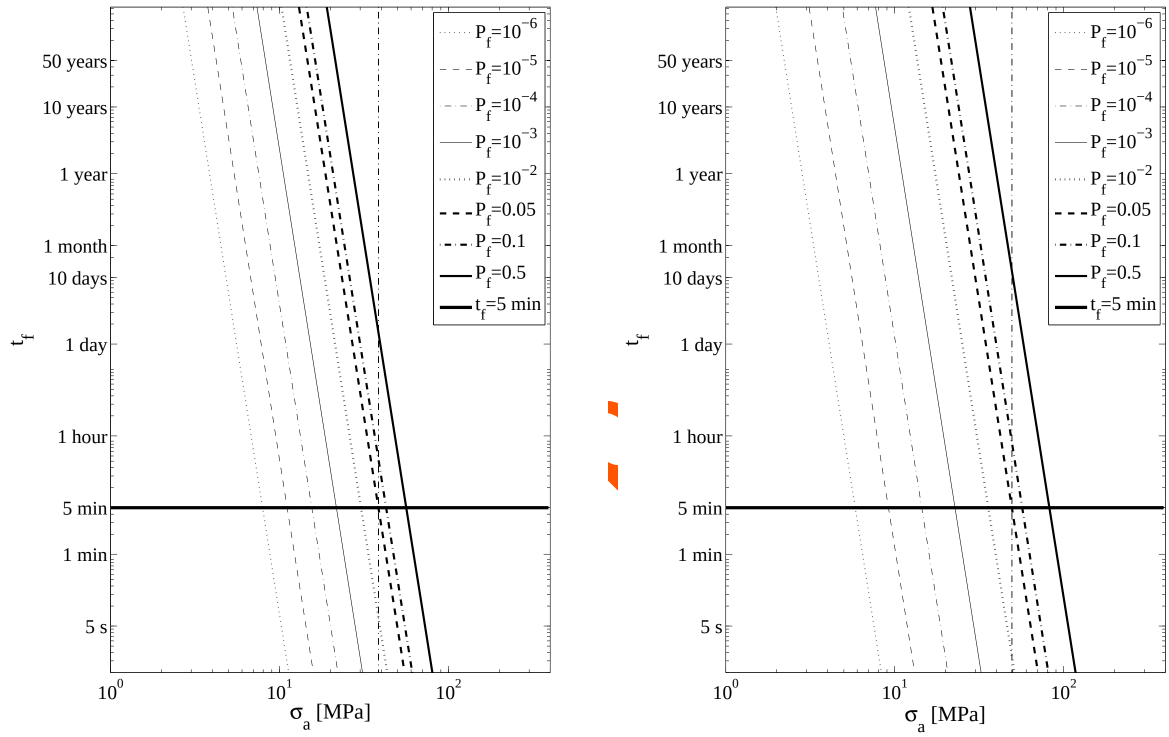
Table 3: Parameters k that define the effective area A_{eff} for the cases under consideration.

Case	k_{tin}	k_{air}
Equibiaxial stress field	1	1
4PB test	0.205	0.237
Plate under uniform pressure	0.138	0.1764

maximum failure stress of 5 minute duration as a function of P_f. At the qualitative level, one can draw the same conclusions already presented for the previous cases.

However, comparisons of figures 1, 4 and 7 evidences that, for fixed load duration, the variation of strength associated with diverse probabilities of failure is not the same. In particular, results for the pressurized plate case are interspersed between the equi-biaxial stress field and the uniaxial stress field of the 4PB case. This is evident from Figure 9, which collects the graphs of Figures 2, 5 and 8: the strength corresponding to a load applied for 5 minutes is plotted as a function of the failure probability (air-side curves are represented with thicker lines). It should be noted that the highest values are obtained for the 4PB test, air-side surface, while the lowest values are in general obtained for an equibiaxial stress field, tin-side surface. All these findings are in agreement with experimental results [4].

The effect of different stress rates $\dot{\sigma}$ on the failure stress f_g has been considered by applying equations (2.28) and (2.29). The corresponding results are plotted in Figure 10. These are qualitatively similar to the graphs of Figures 3 and 6, but one can again notice that the values of strength are, for the case at hand, intermediate between the equi-biaxial and the uniaxial stress state.



(a)

(b)

Figure 7: Plate under uniform pressure. Time-to-failure vs. stress plot for different failure probabilities. a) Tin-side surface; b) air-side surface.

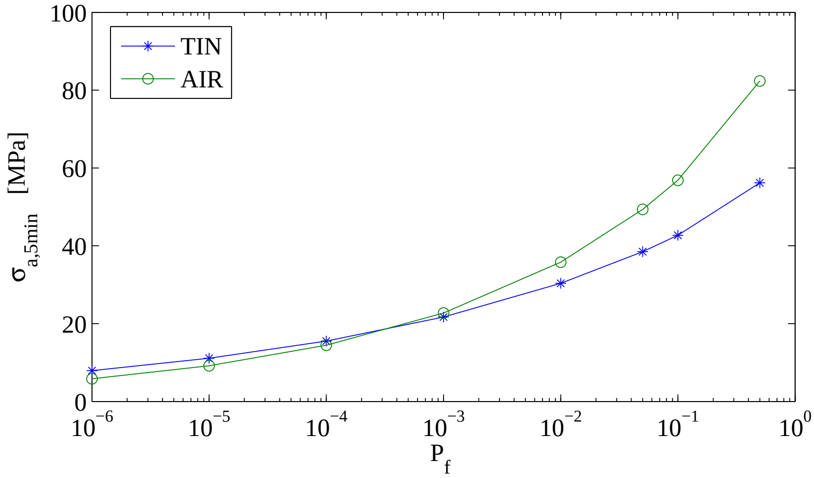


Figure 8: Plate under uniform pressure. Strength corresponding to a load applied for 5 minutes as a function of the failure probability.

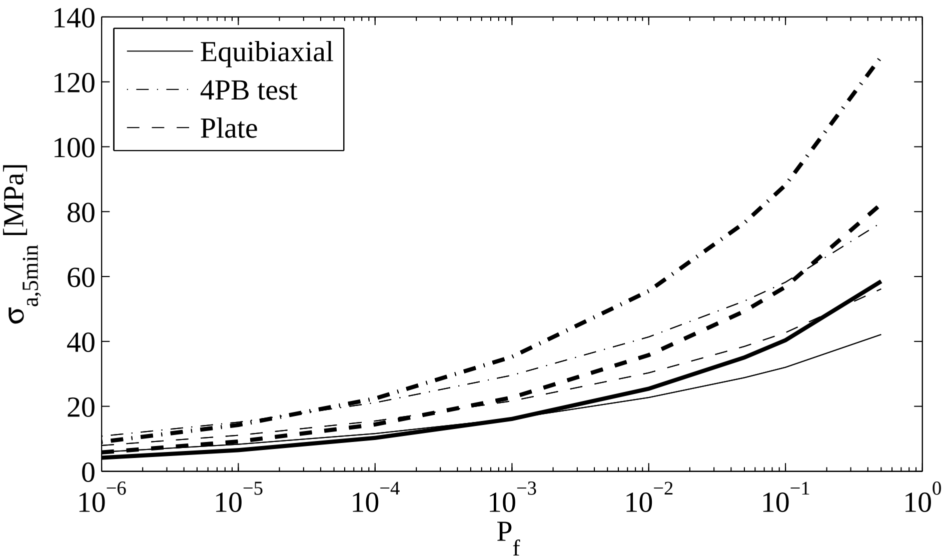


Figure 9: Strength corresponding to a load constantly applied for 5 minutes, as a function of the failure probability. Effect of different stress fields (thin lines refer to the tin-side, while thick lines represent the air-side).

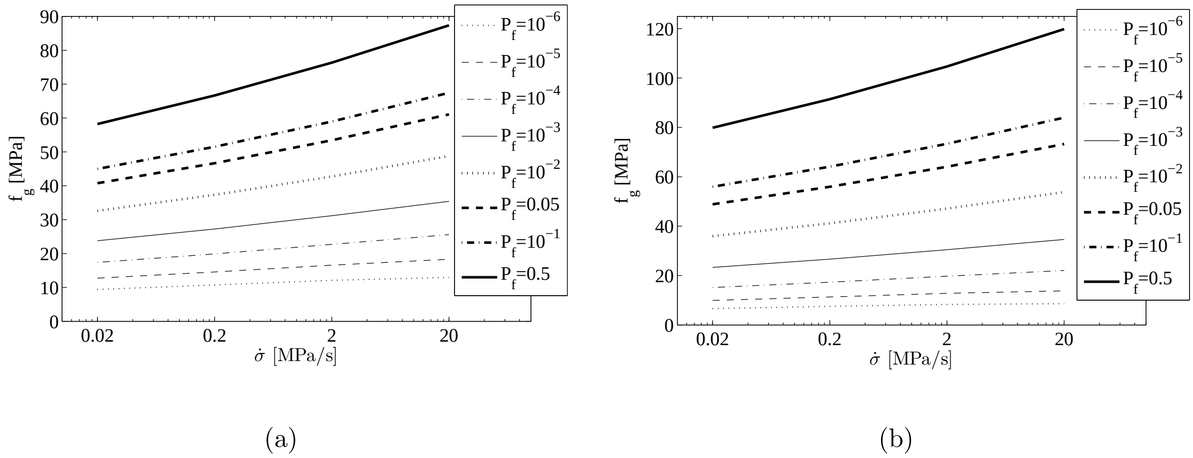


Figure 10: Glass strength as a function of the loading rate for varying failure probabilities. Plate under distributed load: a) tin-side surface; b) air-side surface.

4. Conclusions

The main purpose of this article has been to evidence some critical issues in the design-by-testing of glass structures. Glass is brittle and its strength is governed by the presence of surface flaws, which can open and progress even under constant loading (subcritical crack growth). This causes a strong size-effect, together with the dependence of strength upon the type of loading (uniaxial vs. biaxial) and its duration (static fatigue). Moreover, since there is a great dispersion in the statistical interpretation of the data, the material strength presents a strong dependence upon the associated probability of failure P_f. This means that even a small variation of P_f implies a noteworthy change in the corresponding value of the strength.

Considering a widely accepted model of subcritical crack growth, and interpreting the results from experiments with a distribution *à la* Weibull, it has been possible to analyze in detail the results of hypothetical experiments in three paradigmatic conditions: *i*) plate under an equibiaxial state of stress; *ii*) four-point bending; *iii*) simply supported plate under uniform pressure. If one considers the strength associated with the application of a constant loading for a prescribed time, or the value of strength corresponding to a constant stress rate, a few definitive conclusions can be drawn.

The median value of strength, for which $P_f = 0.5$, can be much higher than the 5% fractile value, which usually represents the reference value in structural standards. However, the difference can be quite variable, depending upon the type of loading and the size of the specimen. For constant loading, the time to failure is strongly dependent upon the target probability, passing from a few minutes to several days if the value of P_f changes from, say, 0.05 to 0.5. On the other hand, for fixed time of loading, a small variation of the applied stress implies a noteworthy variation of the expected probability of failure. This can be of importance if one is interested in accelerating the duration of an experimental campaign. The model proposed here allows to correlate the value of the applied load with the target probability of failure in a prescribed time.

Of course, the type of loading and the size of the specimens strongly affect the results. The failure stress is greater under uniaxial stress field (4PB) than under equi-biaxial stress, because under the latter condition there is the highest probability of finding a dominant crack at right angle to the principal direction of tensile stress. The dependence of the strength upon P_f is also the most marked in an equi-biaxial state of stress. The uniaxial state of stress appears to be associated with a softer dependence, but it is always necessary to carefully consider the state of stress in the whole specimen, calculating in particular the effective area of loading.

Testing at constant load rate has also been considered, evaluating the effects of the load rate on the glass strength as a function of the target failure probability P_f. In such conditions, the ultimate strength is again strongly affected by P_f. In general, the higher the stress rate, the higher is the strength of glass, but this dependence may vary according to P_f. At very low probabilities, of the order of $10^{-6} - 10^{-5}$, such a dependence is barely visible, but at higher probabilities there is a superlinear dependence of the strength upon the logarithm of the stress rate. Also for this loading condition, there is a strong influence of specimen size and type of stress.

Recall that it is always important to distinguish the response of the tin-side of glass from that of the air-side, because they are in general interpreted by two different Weibull distributions.

The air-side presents higher strength than the tin-side, but also higher dispersion of data. Consequently, when considering high target probabilities of failure, of the order of $10^{-3} - 10^{-2}$, the air side appears stronger than the tin-side, but the *vice-versa* is true at lower probabilities, when the effect of dispersion overcomes that of strength.

The three aforementioned examples are quite simple, but all evidence that a careful preliminary analysis is necessary to interpret results from any testing. The size of a full-scale prototype is much larger than the specimens usually employed to assess the material properties according to the current standards; moreover, its state of stress is usually quite complicated, being in general not assimilable to an equi-biaxial or uniaxial state. Therefore, the experimentally-measured glass strength can be highly variable, and such a value does not represent an objective, absolute, indication of the inherent strength, since it has been shown that it depends on the test method as well as on the area under tension. Therefore a probabilistic model, like the one proposed here, has to be used to interpret the results.

Unfortunately our personal experience, deriving from years of revisions of plans on behalf of the Italian Ministry of Infrastructure and Transport, confirms that there are several cases, indeed the majority, in which the reliability of complicated glass structures is claimed from the results of the testing of just one full-scale prototype. Of course, testing one prototype is better than nothing, but it is necessary to take into account all the aforementioned aspects (size effect, type of stress, duration of loading), to determine what is the expected median value of strength for the test according to an assumed statistical distribution of glass strength. Comparison of the value of strength derived from an experiment with the characteristic strength of glass, taken from standards, is meaningless.

Of course, the method proposed here is far from being exhaustive. Perhaps its greatest shortcoming consists in having neglected the effects at the edges, where the defectiveness due to manufacturing is greater than at the center of the glass ply. A specific statistical characterization of the strength close to the borders of a glass ply, possibly interpreted by a Weibull distribution, should be considered to characterize the strength of the prototype. The procedure, which should not neglect to consider the length of the borders (size effect), has been indicated in [19], but the lack of experimental data for this specific aspect does not allow, at the time of the present writing, to derive any quantitative indication. Moreover, no consideration has been made here for heat-strengthened or thermally-toughened glass, but also for this case we are still waiting to have sound experimental data to elaborate.

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